
Fonctions de plusieurs variables

Calculer les dérivées partielles premières et secondes des fonctions suivantes :

1. $f(x, y) = xy^2 - 4x^2 + 3x^3y + 2x - 4$
2. $f(x, y, z) = 2x^3y + 2ye^z + x - 2y^2$
3. $f(x, y) = x^2 \exp(xy)$
4. $f(x, y, z) = x^2y^2\sqrt{z}$
5. $f(x, y, z) = 2y^2(xy + 1) - ze^{-x}$
6. $f(x, y) = \sqrt{x^2 + y^2}$
7. $f(x, y) = \ln(x^3 + y) - 2x$
8. * $f(x, y) = \sin^3(xy) + \cos^2(y)$
9. * $f(x, y) = \ln(x + \sqrt{x^2 + y^2})$
10. $f(x, y, z) = x^2\sqrt{y^2 + z^2}$
11. $f(u, v, w) = \sqrt{u - v} + 2w^2$
12. * $f(x, y) = \frac{x^2y + 3y^3}{1 + x^2 + y^2}$
13. $f(x, y) = \arctan\left(\frac{y}{x}\right)$
14. $f(x, y, z) = \frac{x^3}{yz^2}$
15. * $f(x, y) = \arctan(ye^{-x})$

* : dans cet exercice, les calculs des dérivées partielles secondes peuvent être assez longs

Corrigé

1. $\frac{\partial f}{\partial x}(x, y) = y^2 - 8x + 9x^2y + 2$

$$\frac{\partial f}{\partial y}(x, y) = 2xy + 3x^3,$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -8 + 18xy,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = 2y + 9x^2,$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = 2x.$$

2. $\frac{\partial f}{\partial x}(x, y, z) = 6x^2y + 1,$

$$\frac{\partial f}{\partial y}(x, y, z) = 2x^3 + 2e^z - 4y,$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2ye^z,$$

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = 12xy,$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = -4,$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = 2ye^z,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y, z) = \frac{\partial^2 f}{\partial y \partial x}(x, y, z) = 6x^2,$$

$$\frac{\partial^2 f}{\partial x \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = 0,$$

$$\frac{\partial^2 f}{\partial y \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial y}(x, y, z) = 2e^z.$$

3. $\frac{\partial f}{\partial x}(x, y) = (2x + x^2y) \exp(xy),$

$$\frac{\partial f}{\partial y}(x, y) = x^3 \exp(xy),$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = (2 + 4xy + x^2y^2) \exp(xy),$$

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(x, y) &= \frac{\partial^2 f}{\partial y \partial x}(x, y) \\ &= (3x^2 + x^3y) \exp(xy), \end{aligned}$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = x^4 \exp(xy).$$

4. $\frac{\partial f}{\partial x}(x, y, z) = 2xy^2\sqrt{z},$

$$\frac{\partial f}{\partial y}(x, y, z) = 2yx^2\sqrt{z},$$

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{x^2y^2}{2\sqrt{z}},$$

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = 2y^2\sqrt{z},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = 2x^2\sqrt{z},$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = -\frac{x^2y^2}{4z^{3/2}},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y, z) = \frac{\partial^2 f}{\partial y \partial x}(x, y, z) = 4xy\sqrt{z},$$

$$\frac{\partial^2 f}{\partial x \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = \frac{xy^2}{\sqrt{z}},$$

$$\frac{\partial^2 f}{\partial y \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial y}(x, y, z) = \frac{yx^2}{\sqrt{z}}.$$

5. $\frac{\partial f}{\partial x}(x, y, z) = 2y^3 + ze^{-x},$

$$\frac{\partial f}{\partial y}(x, y, z) = 6y^2x + 4y,$$

$$\frac{\partial f}{\partial z}(x, y, z) = -e^{-x},$$

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = -ze^{-x},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = 12xy + 4,$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y, z) = \frac{\partial^2 f}{\partial y \partial x}(x, y, z) = 6y^2,$$

$$\frac{\partial^2 f}{\partial x \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = e^{-x},$$

$$\frac{\partial^2 f}{\partial y \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial y}(x, y, z) = 0.$$

6. $\frac{\partial f}{\partial x}(x, y) = \frac{x}{\sqrt{x^2 + y^2}},$

$$\frac{\partial f}{\partial y}(x, y) = \frac{y}{\sqrt{x^2 + y^2}},$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{y^2}{(x^2 + y^2)^{3/2}},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = -\frac{xy}{(x^2 + y^2)^{3/2}},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{x^2}{(x^2 + y^2)^{3/2}}.$$

7. $\frac{\partial f}{\partial x}(x, y) = \frac{3x^2}{x^3 + y} - 2,$
 $\frac{\partial f}{\partial y}(x, y) = \frac{1}{x^3 + y},$
 $\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{-3x^4 + 6xy}{(x^3 + y)^2},$
 $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{-3x^2}{(x^3 + y)^2},$
 $\frac{\partial^2 f}{\partial y^2}(x, y) = -\frac{1}{(x^3 + y)^2}.$
8. $\frac{\partial f}{\partial x}(x, y) = 3y \cos(xy) \sin^2(xy),$
 $\frac{\partial f}{\partial y}(x, y) = 3x \cos(xy) \sin^2(xy) - 2 \sin y \cos y,$
 $\frac{\partial^2 f}{\partial x^2}(x, y) = 3y^2 \sin(xy)(2 \cos^2(xy) - \sin^2(xy)),$
 $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = 3 \sin(xy)(\cos(xy) \sin(xy) - xy \sin^2(xy) + 2x^2 \cos^2(xy)),$
 $\frac{\partial^2 f}{\partial y^2}(x, y) = 3x^2 \sin(xy)(2 \cos^2(xy) - \sin^2(xy)) + 2(\sin^2 y - \cos^2 y).$
9. $\frac{\partial f}{\partial x}(x, y) = \frac{\frac{x}{\sqrt{x^2 + y^2}} + 1}{x + \sqrt{x^2 + y^2}} = \frac{1}{\sqrt{x^2 + y^2}},$
 $\frac{\partial f}{\partial y}(x, y) = \frac{y}{x^2 + y^2 + x\sqrt{x^2 + y^2}},$
 $\frac{\partial^2 f}{\partial x^2}(x, y) = -\frac{x}{(x^2 + y^2)^{3/2}},$
 $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = -\frac{y}{(x^2 + y^2)^{3/2}},$
 $\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{x^3 + x^2\sqrt{x^2 + y^2} - y^2\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}(x^2 + y^2 + x\sqrt{x^2 + y^2})^2}.$
10. $\frac{\partial f}{\partial x}(x, y, z) = 2x\sqrt{y^2 + z^2},$
 $\frac{\partial f}{\partial y}(x, y, z) = \frac{x^2 y}{\sqrt{y^2 + z^2}},$
 $\frac{\partial f}{\partial z}(x, y, z) = \frac{x^2 z}{\sqrt{y^2 + z^2}},$
 $\frac{\partial^2 f}{\partial x^2}(x, y, z) = 2\sqrt{y^2 + z^2},$
 $\frac{\partial^2 f}{\partial y^2}(x, y, z) = \frac{x^2 z^2}{(y^2 + z^2)^{3/2}},$
11. $\frac{\partial f}{\partial u}(u, v, w) = \frac{1}{2\sqrt{u - v}},$
 $\frac{\partial f}{\partial v}(u, v, w) = -\frac{1}{2\sqrt{u - v}},$
 $\frac{\partial f}{\partial w}(u, v, w) = 4w,$
 $\frac{\partial^2 f}{\partial u^2}(u, v, w) = -\frac{1}{4(u - v)^{3/2}},$
 $\frac{\partial^2 f}{\partial v^2}(u, v, w) = -\frac{1}{4(u - v)^{3/2}},$
 $\frac{\partial^2 f}{\partial w^2}(u, v, w) = 4,$
 $\frac{\partial^2 f}{\partial u \partial v}(u, v, w) = \frac{\partial^2 f}{\partial v \partial u}(u, v, w) = \frac{1}{4(u - v)^{3/2}},$
 $\frac{\partial^2 f}{\partial u \partial w}(u, v, w) = \frac{\partial^2 f}{\partial w \partial u}(u, v, w) = 0,$
 $\frac{\partial^2 f}{\partial v \partial w}(u, v, w) = \frac{\partial^2 f}{\partial w \partial v}(u, v, w) = 0.$
12. $\frac{\partial f}{\partial x}(x, y) = -\frac{2xy(2y^2 - 1)}{(1 + x^2 + y^2)^2},$
 $\frac{\partial f}{\partial y}(x, y) = \frac{x^2 + x^4 + 8x^2 y^2 + 9y^2 + 3y^4}{(1 + x^2 + y^2)^2},$
 $\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{2y(1 - 3x^2 - y^2 + 6x^2 y^2 - 2y^4)}{(1 + x^2 + y^2)^3},$
 $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = -\frac{2x(-1 - x^2 + 9y^2 + 6x^2 y^2 - 2y^4)}{(1 + x^2 + y^2)^2},$
 $\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{2y(9 + 15x^2 - 3y^2 + 6x^4 - 2x^2 y^2)}{(1 + x^2 + y^2)^2}.$
13. $\frac{\partial f}{\partial x}(x, y) = -\frac{y}{x^2 + y^2},$
 $\frac{\partial f}{\partial y}(x, y) = \frac{x}{x^2 + y^2},$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{2xy}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = -\frac{2xy}{(x^2 + y^2)^2}.$$

$$14. \quad \frac{\partial f}{\partial x}(x, y, z) = \frac{3x^2}{yz^2},$$

$$\frac{\partial f}{\partial y}(x, y, z) = -\frac{x^3}{y^2 z^2},$$

$$\frac{\partial f}{\partial z}(x, y, z) = -\frac{2x^3}{yz^3},$$

$$\frac{\partial^2 f}{\partial x^2}(x, y, z) = \frac{6x}{yz^2},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y, z) = \frac{2x^3}{y^3 z^2},$$

$$\frac{\partial^2 f}{\partial z^2}(x, y, z) = \frac{6x^3}{yz^4},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y, z) = \frac{\partial^2 f}{\partial y \partial x}(x, y, z) = -\frac{3x^2}{y^2 z^2},$$

$$\frac{\partial^2 f}{\partial x \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial x}(x, y, z) = -\frac{6x^2}{yz^3},$$

$$\frac{\partial^2 f}{\partial y \partial z}(x, y, z) = \frac{\partial^2 f}{\partial z \partial y}(x, y, z) = \frac{2x^3}{y^2 z^3}.$$

$$15. \quad \frac{\partial f}{\partial x}(x, y) = \frac{-y e^{-x}}{1 + y^2 e^{-2x}},$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{e^{-x}}{1 + y^2 e^{-2x}},$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = \frac{-y e^{-x}(y^2 e^{-2x} - 1)}{(1 + y^2 e^{-2x})^2},$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{e^{-x}(y^2 e^{-2x} - 1)}{1 + y^2 e^{-2x}},$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = \frac{-2y e^{-3x}}{(1 + y^2 e^{-2x})^2}.$$