TD AL1 : Corrigé des exercices

Exercice 1

Soit $z = a + ib \in \mathbb{C}$ avec $a, b \in \mathbb{R}$.

$$|z| + \bar{z} = 6 - 2i \iff \sqrt{a^2+b^2} + a - ib = 6 - 2i$$
$$\iff \begin{cases} \sqrt{a^2+b^2} + a = 6 \\ -b = -2 \end{cases}$$
$$\iff \begin{cases} \sqrt{a^2+4} + a = 6 \\ b = 2 \end{cases}$$

Il reste à résoudre $\sqrt{a^2+4} + a = 6$

$$\sqrt{a^2+4} + a = 6 \iff \sqrt{a^2+4} = 6 - a$$

- Si $6 - a < 0$ (i.e. $a > 6$) il n'y a pas de solution
- Si $6 - a \geq 0$

$$\sqrt{a^2+4} + a = 6 \iff \sqrt{a^2+4} = 6 - a$$
$$\iff a^2 + 4 = (6-a)^2$$
$$\iff a^2 + 4 = a^2 - 12a + 36$$
$$\iff 12a = 32$$
$$\iff a = \frac{8}{3}$$

(qui est bien ≤ 6)

Conclusion : L'ensemble des solutions est $\left\{ \frac{8}{3} + 2i \right\}$

Exercice 2

1) Soit $z = ib \in i\mathbb{R}$ avec $b \in \mathbb{R}$.

z solution de $(E) \iff (ib)^3 - 2i(ib)^2 - (3-4i)(ib) + 8 + 6i = 0$

$$\iff -ib^3 + 2ib^2 - 3ib - 4b + 8 + 6i = 0$$
$$\iff -4b + 8 + i(-b^3 + 2b^2 - 3b + 6) = 0$$
$$\iff \begin{cases} -4b + 8 = 0 \\ -b^3 + 2b^2 - 3b + 6 = 0 \end{cases}$$

$$\iff \begin{cases} b = 2 \\ -b^3 + 2b^2 - 3b + 6 = 0 \end{cases}$$

$$\iff \begin{cases} b = 2 \\ -8 + 8 - 6 + 6 = 0 \end{cases} \quad \leftarrow \text{vrai}$$

$$\iff z = 2i$$

2) On développe.

3) a) Soit $z = a + ib$ avec $a, b \in \mathbb{R}$.

$\boxed{\Rightarrow}$ Si $z^2 = 3 - 4i$ alors

puisque $z^2 = (a+ib)^2 = a^2 - b^2 + 2iab$

on a : $\begin{cases} a^2 - b^2 = 3 \\ 2ab = -4 \end{cases}$

De plus, $|z|^2 = a^2 + b^2$ et $|z|^2 = |z^2| = |3 - 4i| = 5$

Donc $\begin{cases} a^2 + b^2 = 5 \\ a^2 - b^2 = 3 \\ 2ab = -4 \end{cases}$

$\boxed{\Leftarrow}$ Si $\begin{cases} a^2 + b^2 = 5 \\ a^2 - b^2 = 3 \\ 2ab = -4 \end{cases}$, alors

$$z^2 = a^2 - b^2 + 2iab = 3 - 4i$$

<u>Conclusion</u> : on a bien montré l'équivalence

$$z^2 = 3 - 4i \iff \begin{cases} a^2 + b^2 = 5 \\ a^2 - b^2 = 3 \\ 2ab = -4 \end{cases}$$

b) $\begin{cases} A + B = 5 \\ A - B = 3 \end{cases} \iff \begin{cases} A = 5 - B \\ 5 - 2B = 3 \end{cases} \iff \begin{cases} A = 5 - B \\ B = 1 \end{cases}$

$$\iff \begin{cases} A = 4 \\ B = 1 \end{cases}$$

c) $z^2 = 3-4i \iff \begin{cases} a^2+b^2 = 5 \\ a^2-b^2 = 3 \\ 2ab = -4 \end{cases}$

$\iff \begin{cases} a^2 = 4 \\ b^2 = 1 \\ ab = -2 \end{cases}$

$\iff \begin{cases} a = 2 \quad \text{ou} \quad a = -2 \\ b = 1 \quad \text{ou} \quad b = -1 \\ ab = -2 \end{cases}$

$\iff (a,b) = (2,-1) \quad \text{ou} \quad (a,b) = (-2,1)$

$\iff z = 2-i \quad \text{ou} \quad z = -2+i$

4) Soit $z \in \mathbb{C}$.

z solution de $(E) \iff z - 2i = 0 \quad \text{ou} \quad z^2 = 3-4i$

Donc il y a 3 solutions : $2i$, $2-i$ et $-2+i$

Exercice 3

$z_1 = \sin(\theta) + i\cos(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right) = e^{i(\pi/2 - \theta)}$

$z_2 = 1 + e^{i\pi/4} = e^{i\pi/8}\left(e^{-i\pi/8} + e^{i\pi/8}\right) = e^{i\pi/8} \times \underbrace{2\cos\left(\frac{\pi}{8}\right)}_{>0}$

Exercice 4

1) $\dfrac{z_3}{z_4} = \dfrac{(-1+i\sqrt{3})(-1+i)}{2} = \dfrac{1-\sqrt{3}}{2} - i\,\dfrac{\sqrt{3}+1}{2}$

2) $\dfrac{z_3}{z_4} = \dfrac{2e^{2i\pi/3}}{\sqrt{2}\,e^{-3i\pi/4}} = \sqrt{2}\,e^{17i\pi/12} = \sqrt{2}\,e^{-7i\pi/12}$

$= \sqrt{2}\cos\left(\frac{7\pi}{12}\right) - \sqrt{2}\sin\left(\frac{7\pi}{12}\right)$

3) On en déduit que

$\sqrt{2}\cos\left(\frac{7\pi}{12}\right) = \dfrac{1-\sqrt{3}}{2}$ donc $\cos\left(\frac{7\pi}{12}\right) = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$

$= \dfrac{\sqrt{2}-\sqrt{6}}{4}$

et $\sqrt{2}\sin\left(\frac{7\pi}{12}\right) = \dfrac{\sqrt{3}+1}{2}$ donc $\sin\left(\frac{7\pi}{12}\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Exercice 5

Soit $z \in \mathbb{C}$.

- $z = 0$ n'est pas solution de $(i-z)^4 = z^4$.

  $z = i$ non plus

- Soit $z \in \mathbb{C} \setminus \{0, i\}$.

$$(i-z)^4 = z^4 \iff \left(\frac{i-z}{z}\right)^4 = 1$$

Posons $Z = \dfrac{i-z}{z} \in \mathbb{C}^*$. On peut écrire $Z = re^{i\theta}$ avec $r > 0$ et $\theta \in \mathbb{R}$.

$$\begin{aligned}
(i-z)^4 = z^4 &\iff \left(\frac{i-z}{z}\right)^4 = 1 \\
&\iff Z^4 = 1 \\
&\iff r^4 e^{4i\theta} = 1\, e^{i \times 0} \\
&\iff \begin{cases} r^4 = 1 \\ \exists k \in \mathbb{Z}, \quad 4\theta = 2k\pi \end{cases} \\
&\iff \exists k \in \mathbb{Z}, \quad Z = 1 \times e^{\frac{ki\pi}{2}}.
\end{aligned}$$

Or

$$\begin{aligned}
\frac{i-z}{z} = Z &\iff i - z = Zz \\
&\iff z(Z+1) = i \\
&\iff z = \frac{i}{Z+1} \quad \text{et } Z \neq -1
\end{aligned}$$

Remarquons que $Z = -1$ se produit quand $k = 2, 6, 10, \dots$

Il faut enlever toutes ces possibilités :

$$\begin{aligned}
(i-z)^4 = z^4 &\iff \exists k \in \mathbb{Z}, \quad \frac{i-z}{z} = e^{ki\pi/2} \\
&\iff \exists k \in \mathbb{Z}, \quad z = \frac{i}{e^{ik\pi/2} + 1} \quad \text{et} \quad e^{ik\pi/2} \neq -1
\end{aligned}$$

Il n'y a en fait que 3 solutions :

$k = 0$: $\dfrac{i}{2}$ — $k = 4$, $k = 8$, ...

$k = 1$: $\dfrac{i}{i+1} = \dfrac{i(1-i)}{2} = \dfrac{1-i}{2}$ — $k = 5$, $k = 9$, ...

$k = 2$: impossible — $k = 6$, $k = 10$, ...

$k = 3$: $\dfrac{i}{-i+1} = \dfrac{i(1+i)}{2} = \dfrac{-1+i}{2}$ — $k = 7$, $k = 11$, ...